Pairwise g**-comapct, countably compact, Lindeloff Spaces

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Abstract: In this paper, pairwise g**-comapct space, pairwise g**-countably comapct space, pairwise g**-Lindeloff space are introduced and the relationship between these concepts are studied.

Keywords: *Pairwise g**-comapct space, pairwise g**-countably comapct space, pairwise g**-Lindeloff space*

1. Introduction

Levine **[3]** introduced the class of g-closed sets in 1970 and M.K.R.S. Veerakumar**[10]** introduced g*-closed sets in 1991. We have introduced and studied g**-closed sets **[4]**,g**-compact modulo *I* spaces**[5]**, g**-Lindeloff spaces**[7]** in the year 2012. In this paper pairwise g**-comapct spaces, pairwise g**-countably compact spaces, pairwise g**-Lindeloff spaces are defined and their properties are investigated.

2. Preliminaries

Definition 2.1: A subset A of a topological space(X, τ) is called

1) *generalized closed* (briefly *g-closed*)**[3]** if cl(A) $\subseteq$ U whenever A $\subseteq$ U and U is open in (X, τ).
2) *generalized star closed* (briefly *g*-closed*)**[10]** if cl(A) $\subseteq$ U whenever A $\subseteq$ U and U is g- open in (X, τ).
3) *generalized star star closed* (briefly *g**-closed*)**[4]** if cl(A) $\subseteq$ U whenever A $\subseteq$ U and U is g*- open in (X, τ).

The class of g**-open sets of (X,τ) is denoted by G**O(X, τ)

Definition 2.2:[1] If X is a set and τ_1 and τ_2 are two topologies on X the triple (X, τ_1, τ_2) is defined to be a bitopological space.

Definition 2.3:[2] A cover $\mathcal{U}$ of the topological space (X, τ_1, τ_2) is said to be pairwise open if $\mathcal{U} \subseteq \tau_1 \cup \tau_2$ and $\mathcal{U}$ contains a non empty member of τ_1 and a non empty member of τ_2.

Definition 2.4:[2] If each pairwise open cover has a finite sub cover then the space is said to be pairwise compact.

Definition 2.5:[2] If each pairwise open cover has a countable sub cover then the space is said to be pairwise Lindeloff.

Definition 2.6:[6] A topological space (X, τ) is said to be $g^{**}-multiplicative$ if arbitrary intersection of $g^{**}-closed$ sets is $g^{**}-closed$. Equivalently arbitrary union of $g^{**}-open$ sets is $g^{**}-open$.

Definition 2.7:[7] A topological space (X, τ) is said to be g**-Lindelof if every g**-open cover has a countable sub cover.

Definition 2.8:[5] A collection $\{U_\alpha\}_{\alpha\in\Delta}$ of g**-open sets in X is said to be g**-open cover of X if $X=\bigcup_{\alpha\in\Delta}U_\alpha$

sub collection that also covers X. A subset A of X is said to be g**-compact if every g**-open covering of A contains a finite sub collection that also covers A

Definition 2.10:[5] A subset A of a topological space (X,τ) is said to be g**-countably compact if every countable g**-open covering of A has a finite sub cover.

Definition 2.11:[9]Let $\mathcal{A}\subseteq\wp(X)$. Then $\mathcal{A}$ is said to have finite intersection property (in symbol FIP)if for every finite subfamily $\{A_i\}_{i=1}^{n}$ of $\mathcal{A}$, $\bigcap_{i=1}^{n}A_i\neq\varphi$.

Definition 2.12:[9] Let $\mathcal{A}\subseteq\wp(X)$. Then $\mathcal{A}$ is said to have countable intersection property (in symbol CIP) if for every countable sub family $\{A_i\}_{i=1}^{\infty}$ of $\mathcal{A}$, $\bigcap_{i=1}^{\infty}A_i\neq\varphi$.

3. Pairwise g**-comapct, pairwise g**-countably compact, pairwise g**-Lindeloff Spaces

Definition 3.1: A collection of subsets $\Im$ in the bitopological space (X,τ_1,τ_2) is called pairwise g**-open cover if $X\subseteq\bigcup_{U\in\Im}U$ and $\Im\subseteq G^{**}O(X,\tau_1)\cup G^{**}O(X,\tau_2)$ and $\Im$ contains a non empty member of $G^{**}O(X,\tau_1)$ and a non empty member of $G^{**}O(X,\tau_2)$.

Definition 3.2: A bitopological space (X,τ_1,τ_2) is called pairwise g**-Lindeloff if every pairwise g**-open cover $\Im=\{U_\alpha/\alpha\in\Omega\}$ of X has a countable sub

Definition 2.9:[5]A topological space (X,τ) is said to be g**-compact if every g**-open covering of X contains a finite

collection $\Im\ U_{\alpha_i}, i=1,2,3...............$ of $\Im$

such that $X=\bigcup_{i=1}^{\infty}U_{\alpha_i}$

Definition 3.3: A bitopological space (X,τ_1,τ_2) is called pairwise g**-compact if every pairwise g**-open cover $\Im=\{U_\alpha/\alpha\in\Omega\}$ of X has a countable sub collection $\{U_{\alpha_i}, i=1,2,3............n\}$ of $\Im$ such

that $X=\bigcup_{i=1}^{n}U_{\alpha_i}$

Definition 3.4: A bitopological space (X,τ_1,τ_2) is called pairwise g**-countably compact if every countable pairwise g**-open cover $\Im=\{U_\alpha/\alpha\in\Omega\}$ of X has a finite sub collection $U_{\alpha_i}, i=1,2,3............n$ of $\Im$

such that $X=\bigcup_{i=1}^{n}U_{\alpha_i}$

Theorem 3.5: Every pairwise g**-compact space is pairwise g**-countably compact.
Proof is obvious.

Theorem 3.6: Every pairwise g**-compact space is pairwise g**-Lindeloff.
Proof is obvious.

Theorem 3.7: Every pairwise g**-Lindeloff space is pairwise Lindeloff.
Proof is obvious.

Theorem 3.8: Every pairwise g**-compact space is pairwise compact.
Proof is obvious.

Theorem 3.9: Every pairwise g**-countably compact space is pairwise countably compact.

Proof is obvious.

Theorem 3.10: X is pairwise g**-Lindeloff and pairwise g**-countably compact implies X is pairwise g**- compact.

Proof is obvious.

Theorem 3.11: Let (X,τ_1,τ_2) be a bitopological space. (X,τ_1) and (X,τ_2) be g**-multiplicative. For each non-empty set G in τ_1 and H in τ_2, $\tau_1(H)=\{\varphi, X, V\cup H / V\in G^{**}O(X,\tau_1)\}$ an d $\tau_2(G)=\{\varphi, X, V\cup G / V\in G^{**}O(X,\tau_2)\}$ are topologies in X.

Proof: Let $\{V_\alpha\cup H\}_{\alpha\in\Omega}$ be members of $\tau_1(H)$ then $\underset{\alpha}{\cup}(V_\alpha\cup H)=(\cup V_\alpha)\cup H\in\tau_1(H)$ (since (X,τ_1) is g**-multiplicative).Let $\{V_i\cup H\}_{i=1}^{n}$ be members in $\tau_1(H)$. Then $\overset{n}{\underset{i=1}{\cap}}(V_i\cup H)=(\cap V_i)\cup H\in\tau_1(H)$ Therefore $\tau_1(H)$ is a topology in X. Similarly we can prove $\tau_2(H)$ is a topology in X.

Theorem 3.12: Let (X,τ_1,τ_2) be a bitopological space. (X,τ_1) and (X,τ_2) be finitely g**-additive and g**-multiplicative. Then the following statements are equivalent:

1. (X,τ_1,τ_2) is pairwise g**-compact.
2. For each non empty $G\in G^{**}O(X,\tau_1)$ and $H\in G^{**}O(X,\tau_2)$, $(X,\tau_1(H))$ and $(X,\tau_2(G))$ are compact.
3. Each g**-closed proper subset of (X,τ_1) is g**-compact in (X,τ_2) and each g**-closed proper subset of (X,τ_2) is g**-compact in (X,τ_1).

Proof: $1\Rightarrow 2$: Let $G\in G^{**}O(X,\tau_1)$ and let $\mathcal{A}$ be a open cover for $(X,\tau_2(G))$. Then $\mathcal{A}$ $=\{G\cup U_\alpha / U_\alpha\in G^{**}O(X,\tau_2)\}$. Then

Definition 3.10: Let V be a subset of the bitopological space (X,τ_1,τ_2). Define $\tau_i(V)=\{\varphi, X, U\cup V / U\in\tau_i\}$.

$G\cup\{U_\alpha\}_{\alpha\in\Omega}$ is a pirwise g**-open cover for X. then there exists a finite subcover $G\cup\{\overset{n}{\underset{i=1}{\cup}}U_{\alpha_i}\}$ such that $X=G\cup\{\overset{n}{\underset{i=1}{\cup}}U_{\alpha_i}\}$. Therefore $(X,\tau_2(G))$ is compact. Similar ly $(X,\tau_1(H))$ is compact.

$2\Rightarrow 3$: Let K be a proper g**-closed set in (X,τ_1) then $G=X-K$ is g**-open in (X,τ_1). Let $\{V_\alpha\}$ be a g**-open cover for (X,τ_2) then $\{G\cup V_\alpha\}_{\alpha\in\Omega}$ is a open cover for $(X,\tau_2(G))$. Then there exists $V_{\alpha_1}, V_{\alpha_2}, \ldots\ldots\ldots\ldots.V_{\alpha_n}$ such that $X=\overset{n}{\underset{i=1}{\cup}}(G\cup V_{\alpha_i})=(G\overset{n}{\underset{i=1}{\cup}}V_{\alpha_i})\therefore X-G=\overset{n}{\underset{i=1}{\cup}}V_{\alpha_i}$. That is $K=\overset{n}{\underset{i=1}{\cup}}V_{\alpha_i}$. Therefore K is g**-compact in (X,τ_2).

$3\Rightarrow 1$: Let $\mathcal{A}$ be a pairwise g**-open cover for X. Let τ_1-g**-open set in $\mathcal{A}$ be $\{U_\beta / \beta\in\Omega_1\}$ and τ_2-g**-open set in $\mathcal{A}$ be $\{V_\alpha / \alpha\in\Omega_2\}$.

Case 1: Let $\underset{\alpha\in\Omega_2}{\cup}V_\alpha=X$. Choose $\beta_0\in\Omega_1$ such that $U_{\beta_0}\neq\varphi$. Then $\{V_\alpha / \alpha\in\Omega_2\}$ is a τ_2-g**-open cover for τ_1-g**-closed set $X-U_{\beta_0}$. By (3), there exists $V_{\alpha_1}, V_{\alpha_2}, \ldots\ldots\ldots\ldots.V_{\alpha_n}$ such that $X-U_{\beta_0}=\overset{n}{\underset{i=1}{\cup}}V_{\alpha_i}.\therefore X=\overset{n}{\underset{i=1}{\cup}}V_{\alpha_i}\cup U_{\beta_0}$. Hence (X,τ_1,τ_2) is pairwise g**-compact.

Case 2: Let $\underset{\alpha\in\Omega_2}{\cup}V_\alpha\neq X$. Therefore $K=X-\underset{\alpha\in\Omega_2}{\cup}V_\alpha$ is a τ_2-g**-closed set (since

(X,τ_2) is g**-multiplicative) in X and $K \subseteq \underset{\beta\in\Omega_1}{\cup} U_\beta$. By (3), there exists $U_{\beta_1}, U_{\beta_2}, \ldots\ldots\ldots\ldots U_{\beta_n}$ such that $K = \overset{n}{\underset{i=1}{\cup}} U_{\beta_i}$. Suppose $X = \overset{n}{\underset{i=1}{\cup}} U_{\beta_i}$ then (X,τ_1,τ_2) is pairwise g**-compact. Suppose $X - \overset{n}{\underset{i=1}{\cup}} U_{\beta_i} = \overset{k}{\underset{j=1}{\cup}} V_{\alpha_j.}$ Then $X = \overset{n}{\underset{i=1}{\cup}} U_{\beta_i} \cup \overset{k}{\underset{j=1}{\cup}} V_{\alpha_j.}$ therefore (X,τ_1,τ_2) is pairwise g**-compact.

Theorem 3.13: Let (X,τ_1,τ_2) be a bitopological space. (X,τ_1) and (X,τ_2) be g**-multiplicative. Then the following statements are equivalent:

1. (X,τ_1,τ_2) is pairwise g**-Lindeloff.
2. For each non empty $G \in G^{**}O(X,\tau_1)$ and $H \in G^{**}O(X,\tau_2)$, $(X,\tau_1(H))$ and $(X,\tau_2(G))$ are Lindeloff.
3. Each g**-closed proper subset of (X,τ_1) is g**- Lindeloff in (X,τ_2) and each g**-closed proper subset of is g**- Lindeloff in (X,τ_1).

Proof: Similar to the above proof.

Theorem 3.14: Let (X,τ_1,τ_2) be a bitopological space. (X,τ_1) and (X,τ_2) be g**-multiplicative. Then the following statements are equivalent:

1. (X,τ_1,τ_2) is pairwise g**-countably compact.
2. For each non empty $G \in G^{**}O(X,\tau_1)$ and $H \in G^{**}O(X,\tau_2)$, $(X,\tau_1(H))$ and $(X,\tau_2(G))$ are countably compact.
3. Each g**-closed proper subset of (X,τ_1) is g**- countably compact. in (X,τ_2) and each g**-closed proper

$X \neq UU_{\beta_i}$ then $X - \overset{n}{\underset{i=1}{\cup}} U_{\beta_i}$ is a proper $\tau_1 -$ g**closed set of X contained in $\underset{\alpha\in\Omega_2}{\cup} V_\alpha$. By (3), there exists $V_{\alpha_{1,}} V_{\alpha_{12,}} \ldots\ldots\ldots V_{\alpha_k}$ such that

subset of (X,τ_2) is g**- countably compact. in (X,τ_1).

Proof: Similar to the proof of theorem (3.12).

Theorem 3.15: Let (X,τ_1,τ_2) be a bitopological space. Then the following statements are equivalent:

1. (X,τ_1,τ_2) is pairwise g**- compact.
2. For every family $\{F_\alpha / \alpha \in \Omega\}$ of pairwise g**-closed sets such that $\underset{\alpha\in\Omega}{\cap} F_\alpha = \varphi$, there exists a finite sub family $\{F_{\alpha_i} / i = 1,2,\ldots\ldots\ldots.n\}$ such that $\overset{n}{\underset{i=1}{\cap}} F_{\alpha_i} = \varphi.$
3. For every family $\{F_\alpha / \alpha \in \Omega\}$ of pairwise g**-closed sets with FIP, $\underset{\alpha\in\Omega}{\cap} F_\alpha \neq \varphi.$

Proof: $1 \Rightarrow 2$: Let $\{F_\alpha / \alpha \in \Omega\}$ be a family of pairwise g**-closed sets such that $\underset{\alpha\in\Omega}{\cap} F_\alpha = \varphi$. Then the collection $\{X - F_\alpha / \alpha \in \Omega\}$ is a pairwise g**-open cover of X. by (1), there exists a finite sub collection $\{X - F_{\alpha_i} / i = 1,2\ldots..n\}$ such that $X = \overset{n}{\underset{i=1}{\cup}} (X - F_{\alpha_i})$. Therefore $\overset{n}{\underset{i=1}{\cap}} F_{\alpha_i} = \varphi.$

$2 \Rightarrow 3$: Let Let $\{F_\alpha / \alpha \in \Omega\}$ be a family of pairwise g**-closed sets with FIP. If $\underset{\alpha\in\Omega}{\cap} F_\alpha = \varphi$. By (2), there exists a finite sub family $\{F_{\alpha_i} / i = 1,2,\ldots\ldots\ldots.n\}$ such that

$\bigcap_{i=1}^{n} F_\alpha = \varphi$ which is a contradiction. Therefore $\bigcap_{\alpha\in\Omega} F_\alpha \neq \varphi.$

$3 \Rightarrow 1$: Let $\mathcal{U} = \{U_\alpha / \alpha \in \Omega\}$ be a pairwise g**-open cover of X. To prove $\mathcal{U}$ has a finite sub collection $\{U_{\alpha_i} / i = 1,2........n\}$ such that $X = \bigcup_{i=1}^{n} U_{\alpha_i}$. If not, then $\{X - U_\alpha / \alpha \in \Omega\}$ is a family of pairwise g**-closed sets such that $\bigcap_{i=1}^{n} \{X - U_\alpha\} \neq \varphi$ for any finite subset of Ω. By (3), $\bigcap_{\alpha\in\Omega} \{X - U_\alpha\} \neq \varphi.$ $\therefore X \neq \bigcup_{\alpha\in\Omega} U_\alpha$ which is a contradiction. therefore $\mathcal{U}$ has a finite sub cover. Hence (X, τ_1, τ_2) is pairwise g**-compact.

Theorem 3.16: Let (X, τ_1, τ_2) be a bitopological space. Then the following statements are equivalent:

1. (X, τ_1, τ_2) is pairwise g**- Lindeloff.
2. For every family $\{F_\alpha / \alpha \in \Omega\}$ of pairwise g**-closed sets such that $\bigcap_{\alpha\in\Omega} F_\alpha = \varphi$, there exists a countable sub family $\{F_{\alpha_i} / i = 1,2,.........n\}$ such that $\bigcap_{i=1}^{n} F_{\alpha_i} = \varphi.$
3. For every family $\{F_\alpha / \alpha \in \Omega\}$ of pairwise g**-closed sets with CIP, $\bigcap_{\alpha\in\Omega} F_\alpha \neq \varphi.$

Proof: Similar to the above proof.

Theorem 3.17: Let (X, τ_1, τ_2) be a bitopological space. Then the following statements are equivalent:

1. (X, τ_1, τ_2) is pairwise g**- countably compact.
2. For every countable family $\{F_i / i = 1,2,........\}$ of pairwise g**-closed sets such that $\bigcap_{i=1}^{\infty} F_\alpha = \varphi$, there exists a finite sub family $\{F_{\alpha_i} / i = 1,2,.........n\}$ such that $\bigcap_{i=1}^{n} F_{\alpha_i} = \varphi.$
3. For every countable family $\{F_i / i = 1,2,......\}$ of pairwise g**-closed sets with CIP, $\bigcap_{i=1}^{\infty} F_\alpha \neq \varphi.$

Proof: Similar to the proof of theorem(3.15).

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